

Free massless particles and extended space-time algebra

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February 1, 2008

Abstract

We study the space-time invariances of the relativistic particle action for both the massive and massless cases. While the massive action has only the invariances associated to the Poincaré algebra, we find that the invariances of the massless action give rise to the conformal algebra in four dimensions. For the free massless particle, a new invariance of the action permits the construction of an extension of the conformal algebra. The conclusion is that two distinct symmetry breaking mechanisms are necessary to arrive at the Poincaré algebra.

PACS numbers: 11.15.-q, 11.15.Kc, 11.25.Hf

1 Introduction

The relativistic particle continues to be one of the most interesting dynamical systems to investigate if one wishes to understand fundamental physics. This is because relativistic particle theory has many features that have higher-dimensional analogues in relativistic string theory, while, at the same time, being a prototype of general relativity. In the well-known “einbein” version [1], the action for the relativistic particle

$$S = \frac{1}{2} \int d\tau (e^{-1} \dot{x}^2 - em^2) \quad (1)$$

defines a generally covariant one-dimensional field theory where the particle mass m plays the role of a cosmological constant [2]. Action (1) is then the simplest theoretical laboratory where we can investigate the origin of a non-vanishing cosmological constant [3].

The classical equation of motion for x^μ that follows from Hamilton’s principle applied to action (1) is

$$\frac{d}{d\tau} (e^{-1} \dot{x}^\mu) = 0 \quad (2)$$

and the particle will be free only when the condition $\frac{de}{d\tau} = 0$ is satisfied. As the variable $e(\tau)$ is associated to the geometry of the particle world-line, this condition is equivalent to a constant, non-dynamical, geometry. Let us study the space-time invariances of action (1). It is invariant under Poincaré transformations

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \quad (3a)$$

$$\delta e = 0 \quad (3b)$$

and under the diffeomorphisms

$$\delta x^\mu = \dot{\epsilon} x^\mu \quad (4a)$$

$$\delta e = \frac{d}{d\tau}(\epsilon e) \quad (4b)$$

In consequence of the invariance of action (1) under the Poincaré transformation (3), the following vector field is defined on the background space-time [4]

$$V = a^\mu P_\mu - \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} \quad (5)$$

where

$$P_\mu = \partial_\mu \quad (6)$$

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \quad (7)$$

$a^\mu P_\mu$ is the field of space-time translations and $\frac{1}{2} \omega^{\mu\nu} M_{\mu\nu}$ is the field of space-time rotations. The generators of these fields obey the algebra

$$[P_\mu, P_\nu] = 0 \quad (8a)$$

$$[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu \quad (8b)$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \quad (8c)$$

The algebra (8) is the Poincaré space-time algebra.

2 Massless particles

Let us now consider the massless particle action

$$S = \frac{1}{2} \int d\tau e^{-1} \dot{x}^2 \quad (9)$$

which is the $m = 0$ limit of action (1). This action is also invariant under the Poincaré transformations (3) and under the diffeomorphisms (4). The equation

of motion for x^μ that follows from (9) is identical to (2), and the massless particle will only be free if the one-dimensional geometry defined by the particle world-line is a non-dynamical one.

The massless action (9) has a larger set of space-time invariances. It is also invariant under the scale transformation

$$\delta x^\mu = \alpha x^\mu \quad (10a)$$

$$\delta e = 2\alpha e \quad (10b)$$

where α is a constant, and under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \eta^{\mu\nu} x^2) b_\nu \quad (11a)$$

$$\delta e = 4e x \cdot b \quad (12b)$$

where b_μ is a constant vector. As a consequence of the invariances of the massless action we can define the space-time vector field [4]

$$V_0 = a^\mu P_\mu - \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} + \alpha D + b^\mu K_\mu \quad (13)$$

where

$$D = x^\mu \partial_\mu \quad (14)$$

and

$$K_\mu = (2x_\mu x^\nu - \delta_\mu^\nu x^2) \partial_\nu \quad (15)$$

The two additional vector fields on the right of equation (13) are respectively associated with dilatations and conformal boosts. The generators of the vector field V_0 obey the algebra

$$[P_\mu, P_\nu] = 0 \quad (16a)$$

$$[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu \quad (16b)$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \quad (16c)$$

$$[D, D] = 0 \quad (16d)$$

$$[D, P_\mu] = -P_\mu \quad (16e)$$

$$[D, M_{\mu\nu}] = 0 \quad (16f)$$

$$[D, K_\mu] = K_\mu \quad (16g)$$

$$[P_\mu, K_\nu] = 2(\delta_{\mu\nu}D - M_{\mu\nu}) \quad (16h)$$

$$[M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda}K_\mu - \delta_{\lambda\mu}K_\nu \quad (16i)$$

$$[K_\mu, K_\nu] = 0 \quad (16j)$$

This is the conformal algebra in a $D = 4$ space-time [4], the extension of the Poincaré algebra (8).

Let us now restrict the analysis to the case of a non-dynamical one-dimensional geometry. Using the equation of free motion, it can be verified that the massless particle action (9) is also invariant under the transformation [5]

$$x^\mu \rightarrow \exp\left\{\frac{1}{3}\beta(\dot{x}^2)\right\}x^\mu \quad (17a)$$

$$e \rightarrow \exp\left\{\frac{2}{3}\beta(\dot{x}^2)\right\}e \quad (17b)$$

where β is an arbitrary function of \dot{x}^2 . This symmetry is interesting because it has a higher-dimensional extension in the tensionless limit of string theory [5]. Just as the massless limit is the high-energy limit of particle theory, the tensionless limit [6] is the high-energy limit [7] of string theory. Here, if τ is taken to be the particle's proper time, then \dot{x}^μ is the four-velocity and equations (17) define a scale transformation that depends on the square of the four-velocity. Infinitesimally we can then define velocity-dependent scale transformations

$$\delta x^\mu = \alpha\beta(\dot{x}^2)x^\mu \quad (18)$$

where α is the same constant that appears in equations (10). These transformations then lead to the existence of velocity-dependent dilatations. The vector field D of equation (14) can then be changed according to

$$D = x^\mu \partial_\mu \rightarrow D^* = x^\mu \partial_\mu + \beta(\dot{x}^2)x^\mu \partial_\mu \quad (19)$$

Because all vector fields in equation (13) involve partial derivatives with respect to x^μ and β is a function of \dot{x}^μ , we can also introduce the generators

$$P_\mu^* = P_\mu + \beta P_\mu \quad (20)$$

$$M_{\mu\nu}^* = M_{\mu\nu} + \beta M_{\mu\nu} \quad (21)$$

$$K_\mu^* = K_\mu + \beta K_\mu \quad (22)$$

and define a new vector field V_0^* by

$$V_0^* = a^\mu P_\mu^* - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu}^* + \alpha D^* + b^\mu K_\mu^* \quad (23)$$

The generators of this vector field obey the algebra

$$[P_\mu^*, P_\nu^*] = 0 \quad (24a)$$

$$[P_\mu^*, M_{\nu\lambda}^*] = (\delta_{\mu\nu}P_\lambda^* - \delta_{\mu\lambda}P_\nu^*) + \beta(\delta_{\mu\nu}P_\lambda^* - \delta_{\mu\lambda}P_\nu^*) \quad (24b)$$

$$\begin{aligned} [M_{\mu\nu}^*, M_{\lambda\rho}^*] = & (\delta_{\nu\lambda}M_{\mu\rho}^* + \delta_{\mu\rho}M_{\nu\lambda}^* - \delta_{\nu\rho}M_{\mu\lambda}^* - \delta_{\mu\lambda}M_{\nu\rho}^*) \\ & + \beta(\delta_{\nu\lambda}M_{\mu\rho}^* + \delta_{\mu\rho}M_{\nu\lambda}^* - \delta_{\nu\rho}M_{\mu\lambda}^* - \delta_{\mu\lambda}M_{\nu\rho}^*) \end{aligned} \quad (24c)$$

$$[D^*, D^*] = 0 \quad (24d)$$

$$[D^*, P_\mu^*] = -P_\mu^* - \beta P_\mu^* \quad (24e)$$

$$[D^*, M_{\mu\nu}^*] = 0 \quad (24f)$$

$$[D^*, K_\mu^*] = K_\mu^* + \beta K_\mu^* \quad (24g)$$

$$[P_\mu^*, K_\nu^*] = 2(\delta_{\mu\nu}D^* - M_{\mu\nu}^*) + 2\beta(\delta_{\mu\nu}D^* - M_{\mu\nu}^*) \quad (24h)$$

$$[M_{\mu\nu}^*, K_\lambda^*] = (\delta_{\lambda\nu}K_\mu^* - \delta_{\lambda\mu}K_\nu^*) + \beta(\delta_{\lambda\nu}K_\mu^* - \delta_{\lambda\mu}K_\nu^*) \quad (24i)$$

$$[K_\mu^*, K_\nu^*] = 0 \quad (24j)$$

Notice that the vanishing brackets of the conformal algebra (16) are preserved as vanishing in the above algebra, but the non-vanishing brackets of the conformal algebra now have linear and quadratic contributions from the arbitrary function $\beta(\dot{x}^2)$. It is interesting to choose β simply linear in \dot{x}^2 because then, if the classical equation of motion for $e(\tau)$ that follows from the massless action (9) is imposed, the transformation (17) becomes the identity transformation. The algebra (24) is then on-shell in x^μ but off-shell in e .

The conclusions of this letter are the following: For the relativistic particle, two different kinds of symmetry breaking mechanisms are necessary to go from the algebra (24) to the Poincaré algebra (8). The first one is the appearance of an interaction. When this occurs, the free motion equation is no longer valid and the invariance (17) disappears. The algebra (24) then reduces to the conformal algebra (16). The second mechanism must generate a non-vanishing particle mass. This will destroy scale invariance and conformal invariance, and the conformal algebra (16) finally reduces to the Poincaré algebra (8). These features of relativistic particle theory may have some significance in relation to the cosmological constant problem because, as we mentioned in the introduction, the particle mass plays the role of a cosmological constant in the one-dimensional generally covariant field theory defined by action (1). According to the results of this letter, a non-vanishing cosmological constant may be the result of symmetry breaking mechanisms.

References

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